GENERALIZED SIMILARITY OF LAMINAR NONISOTHERMIC FLOWS OF A VISCOUS GAS IN THIN PIPES OF VARIABLE CROSS SECTION

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The method of generalized similarity of the boundary-layer theory is applied to the problem of laminar flow of a viscous gas in thin circular pipe with cross section which varies along the length of the pipe.

GENERAL FORMULATION OF THE PROBLEM

We consider the steady-state laminar flow of a viscous gas in a circular thin heated pipe with radius $r_0(x)$ which varies smoothly from one cross section to another, and with temperature T(x). We use the generally accepted assumptions of the dynamics of viscous gases [1]: The gas is an ideal Newtonian medium; the dynamic viscosity coefficient μ is a power function of temperature; the specific heats c_p and c_v, their ratio $\varkappa = c_p/c_v$, and the Prandtl number $\mathrm{Pr} = \mu c_p/\lambda$ are temperature-independent, and are physical constants of the gas. Under these assumptions, using the "thin layer" approximation in the variables of Dorodnitsyn [1, 2]

$$\xi = x;\ \eta = \int_0^r \frac{\rho}{\rho_w} dr \tag{1}$$

the gas flow is described by the system of equations

$$\frac{\partial ru}{\partial \xi} + \frac{\partial r\tilde{v}}{\partial \eta} = 0,\ u\frac{\partial u}{\partial \xi} + \tilde{v}\frac{\partial u}{\partial \eta} = -\frac{1}{\rho}\frac{dp}{d\xi} + \frac{\nu_w}{r}\frac{\partial}{\partial \eta} H^{n-1} r\frac{\partial u}{\partial \eta},$$
$$u\frac{\partial h}{\partial \xi} + \tilde{v}\frac{\partial h}{\partial \eta} = \frac{u}{\rho}\frac{dp}{d\xi} + \nu_w H^{n-1}\left(\frac{\partial u}{\partial \eta}\right)^2 + \frac{1}{\mathrm{Pr}}\frac{\nu_w}{r}\frac{\partial}{\partial \eta} H^{n-1} r\frac{\partial h}{\partial \eta} \tag{2}$$

or, if we introduce the flow function ψ according to the formulas

$$ru = \frac{\partial \psi}{\partial \eta},\ r\tilde{v} = -\frac{\partial \psi}{\partial \xi}, \tag{3}$$

the flow is described by the system

$$\frac{\partial \psi}{\partial \eta}\frac{\partial^2 \psi}{\partial \xi \partial \eta} - \frac{\partial \ln r}{\partial \xi}\left(\frac{\partial \psi}{\partial \eta}\right)^2 - \frac{\partial \psi}{\partial \xi}\frac{\partial^2 \psi}{\partial \eta^2} + \frac{\partial \ln r}{\partial \eta}\frac{\partial \psi}{\partial \eta}\frac{\partial \psi}{\partial \xi} =$$
$$= -\frac{r^2}{\rho}\frac{dp}{d\xi} + r\nu_w\frac{\partial}{\partial \eta}\left\{H^{n-1}\left[\frac{\partial^2 \psi}{\partial \eta^2} - \frac{\partial \ln r}{\partial \eta}\frac{\partial \psi}{\partial \eta}\right]\right\},$$
$$\frac{\partial \psi}{\partial \eta}\frac{\partial h}{\partial \xi} - \frac{\partial \psi}{\partial \xi}\frac{\partial h}{\partial \eta} = \frac{1}{\rho}\frac{\partial \psi}{\partial \eta}\frac{dp}{d\xi} + r\nu_w H^{n-1}\left[\frac{\partial}{\partial \eta}\left(\frac{1}{r}\frac{\partial \psi}{\partial \eta}\right)\right] + \frac{1}{\mathrm{Pr}}\nu_w\frac{\partial}{\partial \eta} H^{n-1} r\frac{\partial h}{\partial \eta} \tag{4}$$

with the conditions

$$\psi = 0,\ \partial\psi/\partial\eta = 0 \text{ for } \eta = 0;\ \psi = Q,\ \partial\psi/\partial\eta = 0 \text{ for } \eta = \Delta(\xi); \tag{5}$$
$$\psi = \psi(\xi_0, \eta) \text{ for } \xi = \xi_0.$$

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UNIVERSAL EQUATIONS

To integrate the system of equations (4) with conditions (5), we use the ideas of the method of generalized similarity of the boundary-layer theory of Loitsyanskii [1], applied to the laminar flows of a viscous fluid in a thin layer in [3]. The essence of the method consists of writing Eq. (4) and conditions (5) in terms of variables which include kinematic, geometric, and thermodynamic conditions of the problem in such a way that the final form of the equations does not contain these conditions. Consequently, these equations and their solution will be universal for all problems of the same class.

We shall show that these equations can indeed be constructed. To this end, we replace the variables in (4) and (5) by the coordinate φ and by reduced flow function and enthalpy Φ and H as follows:

$$\varphi = \frac{\eta}{\Delta(\xi)}; \quad \psi(\xi, \eta) = Q\Phi(\xi, \varphi); \quad h(\xi, \eta) = h_w(\xi) H(\xi, \varphi). \tag{6}$$

As a result of this transformation, the system and conditions take on the following form (the dot denotes differentiation with respect to φ, and prime with respect to ξ):

$$\frac{\partial}{\partial\varphi}\left\{H^{n-1}\left[\ddot{\Phi} - \dot{\Phi}\frac{\partial \ln \bar{r}}{\partial\varphi}\right]\right\} - \frac{Q\cdot\Delta'}{\nu_w\Delta}\frac{1}{r}\dot{\Phi}^2 \frac{Q}{\nu_w}\frac{1}{r}\left\{\ddot{\Phi}\frac{\partial\Phi}{\partial\xi} - \dot{\Phi}\frac{\partial\dot{\Phi}}{\partial\xi} + \dot{\Phi}\left[\dot{\Phi}\frac{\partial \ln \bar{r}}{\partial\xi} - \frac{\partial \ln \bar{r}}{\partial\varphi}\frac{\partial\Phi}{\partial\xi}\right]\right\} = \bar{r}pH, \tag{7}$$

$$\frac{Q}{\nu_w}\frac{h_w'}{h_w}\dot{\Phi}H + \frac{Q}{\nu_w}\left[\frac{\partial H}{\partial\xi}\dot{\Phi} - \dot{H}\frac{\partial\Phi}{\partial\xi}\right] = \frac{Q^2}{h_w\cdot\Delta^4}\left\{pH + H^{n-1}\left[\ddot{\Phi} - \frac{\partial \ln \bar{r}}{\partial\varphi}\dot{\Phi}\right]^2\right\} + \frac{1}{\mathrm{Pr}}\frac{\partial}{\partial\varphi}\bar{r}H^{n-1}\frac{\partial H}{\partial\varphi},$$

$$\Phi = 0, \ \dot{\Phi} = 0 \ \text{for} \ \varphi = 0; \ \Phi = 1, \ \dot{\Phi} = 0 \ \text{for} \ \varphi = 1. \tag{8}$$

The equations contain dimensionless complexes which give the basis for construction of sets or series of these complexes, i.e., form-parameters or simply parameters:

$$f_k = \frac{Q^k}{\nu_w^k}\frac{d^k\Delta}{d\xi^k}\frac{1}{\Delta}, \quad t_k = \frac{Q^k}{\nu_w^k}\frac{d^k h_w}{d\xi^k} 1/h_w, \quad k = 0, 1, 2\ldots \tag{9}$$

These parameters reflect the effect of geometry and surface temperature distribution on the formation of velocity and temperature profiles in the cross section of pipe. In addition, the heat-balance equation contains the complex

$$e = Q^2/h_w\Delta^4, \tag{10}$$

which is the analogue of the Ekkert number of the heat-exchange theory [2].

The derivatives with respect to ξ of the series of parameters (9) give the recursion relations

$$\frac{Q}{\nu_w}\frac{df_k}{d\xi} = f_{k+1} - (k\omega + f_1)f_k = Q_{k+1}, \quad \frac{Q}{\nu_w}\frac{dt_k}{d\xi} = t_{k+1} - (k\omega + t)t_k = \gamma_{k+1}, \tag{11}$$

which relate the parameters with consecutive numbers and form the basis for obtaining the series of parameters (9). The derivative with respect to the parameter e

$$\frac{Q}{\nu_w}\frac{de}{d\xi} = -(4f_1 + t_1)e \tag{12}$$

can be expressed in terms of this parameter, f_1, and t_1 in such a way that it does not form a new series. In expression (11), ω denotes a parameter which contains a derivative of ν_w and has the following form:

$$\frac{Q}{\nu_w^2}\frac{d\nu_w}{d\xi} = -\frac{\varkappa}{\varkappa - 1}ep - (n+1)t_1. \tag{13}$$

Since the pressure is independent of the transverse coordinate, the reduced pressure gradient

$$P = \frac{\Delta^4}{\mu_w Q}\frac{dp}{d\xi} \tag{14}$$

in Eqs. (7) and (13) is a function of only f_k, t_k, and e, and can be expressed in terms of these quantities using the condition $\Phi(f_k, t_k, e, 1) = 1$.

The definition of the parameters contains functions which are continuous and can be differentiated any number of times. These functions which are arbitrary can be regarded as mutually independent, and can be considered as new variables. Using the formula (summation with respect to k)

$$\frac{Q}{\nu_w}\frac{\partial}{\partial \xi} = Q_{k+1}\frac{\partial}{\partial f_k} + \gamma_{k+1}\frac{\partial}{\partial t_k} - (4f_1 + t_1)\,e\,\frac{\partial}{\partial e} = L \tag{15}$$

one transforms from the variable ξ to the functional space of parameters $\{f_k, t_k e\}$. All this is done in such a way that Eq. (7) can be written in the universal form (in the above sense) which does not contain the concrete features of the problem:

$$\frac{\partial}{\partial\varphi}\left\{H^{n-1}\left[\ddot{\Phi} - \dot{\Phi}\frac{\partial \ln \bar{r}}{\partial\varphi}\right]\right\} + f_1\frac{1}{r}\dot{\Phi}^2 + \frac{1}{r}\left\{\ddot{\Phi}L(\Phi) - \dot{\Phi}L(\dot{\Phi}) + \dot{\Phi}\left[\dot{\Phi}L(\ln\bar{r}) - \frac{\partial \ln r}{\partial\varphi}L(\Phi)\right]\right\} = \bar{r}pH,$$

$$t_1\dot{\Phi}H + [\dot{\Phi}L(H) - \dot{H}L(\Phi)] = e\left\{pH + H^{n-1}\left[\ddot{\Phi} - \frac{\partial \ln \bar{r}}{\partial\varphi}\dot{\Phi}\right]^2\right\} + \frac{1}{p_r}\frac{\partial}{\partial\varphi}\bar{r}H^{n-1}\frac{\partial H}{\partial\varphi} \tag{16}$$

with conditions (8). The last condition in (5) is replaced by one which follows from Eqs. (16) if the parameters of generalized similarity are set equal to zero.

SIMILARITY OF THE FLOW OF VISCOUS GAS IN PIPES. PARAMETRIC APPROXIMATIONS

Since the number of parameters is infinite, it is impossible to integrate Eq. (16) obtained above. In practice, one therefore limits oneself to a small number of parameters, and integrates "finite fragments" of these equations.

Keeping f_1, t_1, and e in the equations, and omitting the operators of differentiation with respect to the parameters, we have the system of equations

$$\frac{\partial}{\partial\varphi}\left\{H^{n-1}\left[\ddot{\Phi} - \dot{\Phi}\frac{\partial \ln \bar{r}}{\partial\varphi}\right]\right\} + f_1\frac{1}{r}\dot{\Phi}^2 = \bar{r}pH;$$

$$t\dot{\Phi}H = e\left\{pH + H^{n-1}\left[\ddot{\Phi} - \dot{\Phi}\frac{\partial \ln \bar{r}}{\partial\varphi}\right]^2\right\} + \frac{1}{\mathrm{Pr}}\frac{\partial}{\partial\varphi}\bar{r}H^{n-1}\frac{\partial H}{\partial\varphi} \tag{17}$$

with conditions (8). When the parameters are independent of the cross section, Eqs. (17) express the strict similarity, or automodelling, of the flow in pipes with variable cross section. When the parameters vary, the equations give a simpler similarity concept, i.e., local similarity with respect to these parameters which, for $f_2 = t_2 = 0$, has a meaning as long as $(\omega + f_1)f_1$ or $(\omega + t_1)t_1$ are small, i.e., the derivatives with respect to parameters can be neglected. When these quantities vary considerably, one needs to include the derivatives with respect to the parameters f_1 and t_1 in the operator L. This is the so-called parametric approximations. On the example of the equation of motion, this means the following. The one-parameter approximation ($f_1 \neq 0$; $f_2 = f_3 \ldots f_k = 0$) is the fragment of equation of the type

$$\frac{\partial}{\partial\varphi}\left\{H^{n-1}\left[\ddot{\Phi} - \dot{\Phi}\frac{\partial \ln \bar{r}}{\partial\varphi}\right]\right\} - \frac{1}{f_1}\frac{1}{r}\dot{\Phi}^2 - (\omega + f_1)f_1\left\{\ddot{\Phi}\frac{\partial\Phi}{\partial f_1} - \dot{\Phi}\frac{\partial\dot{\Phi}}{\partial f_1} + \dot{\Phi}\left[\dot{\Phi}\frac{\partial \ln \bar{r}}{\partial\varphi} - \frac{\partial \ln \bar{r}}{\partial\varphi}\frac{\partial\Phi}{\partial f_1}\right]\right\} = \bar{r}PH. \tag{18}$$

Equation (18) is exact if $f_2 = 0$, i.e., when the function $\Delta(\xi)$ is linear. In general, the n-parameter approximation is exact if this function is a polynomial of order n. One can construct also intermediate or local n-parameter approximations, by keeping n parameters f_k (k = 1, 2, … n) in the equations but omitting the derivative with respect to the parameter f_n. In an analogous fashion, one can solve the problem by approximations with respect to the parameters t_k.

Thus, it is suggested that the problem of flow of viscous gas in pipes of variable cross section is solved in two stages. In the first stage, one solves once and for all Eq. (16) with conditions (8) in some parametric approximation. One also determines the functions $\Phi(\varphi; f_1 \ldots f_n; t_1 \ldots t_n; e)$; $H(\varphi; f_1 \ldots f_n; t_1 \ldots t_n; e)$; $P(f_1 \ldots f_n; t_1 \ldots t_n; e)$, so that the reduced friction coefficient is $\xi = \ddot{\Phi}(1)$, and the heat flux is $q = \dot{H}(1)$. In the second stage, one calculates the pressure distribution and other characteristics of the flow from the concrete laws of variation of the pipe radius and temperature. To this end, the results of the first stage must be used to obtain an explicit form of functions of the parameters, and the Dorodnitsyn variables (1) must be replaced by the "physical" variables x and r.

We note that the theoretical analysis of similarity of flow of a viscous gas in pipes with heat exchange is analyzed theoretically, e.g., in [4].

NOTATION

x, r, cylindrical coordinates; u, v, velocity components; p_W, ρ_W, and ν_W, pressure, density, and kinematic coefficient of viscosity for temperature of the wall; $\bar{v} = v \cdot H^{-1} + \frac{\partial \eta}{\partial x} u$; $\Delta = \int_0^{r_0(x)} H^{-1} dr$; $\bar{r} = \int_0^{\varphi} H d\varphi$; $Q = \int_0^{\Delta} r u d\eta = \frac{Q_0}{2\pi \rho_w}$; Q_0, constant flow rate of the gas.

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CALCULATION OF THE LAMINAR FLOW OF AN INCOMPRESSIBLE LIQUID AROUND A DISC AND A CYLINDER

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We study the circular flow of a viscous incompressible liquid around a disc and a cylinder, in the range of Reynolds numbers $40 \leq \text{Re} \leq 1000$.

The aim of the present work is to obtain stable and sufficiently accurate numerical solutions for the flow near a disc and a cylinder. The bodies are immersed in a circular flow of a viscous incompressible liquid, with a zero angle of incidence. This type of information is essential since, in the construction of models of flow of a liquid which contains solid particles, one usually uses the data about the action of the liquid on an individual particle [1]. The solution is based on the difference approximation of the Navier–Stokes equations according to a scheme used in [2]. There are a number of features of the scheme that make it useful for the study of the flow considered here, which is characterized by the presence of developed circulation zones. These features are: the use of velocity components and pressure correction as the basic independent variables, the displacement of the grid for velocity components, the combination of unilateral and central difference in the approximation of convection terms (hybrid scheme). The solution is limited to the region of Reynolds numbers constructed from the unperturbed flow and from the diameter of the disc or cylinder, i.e., $40 \leq \text{Re} \leq 1000$. Possible effects of three-dimensionality of the flow were not considered.

In a cylindrical coordinate system (x, r), the equation for the change of momentum and the continuity equation can be written in the form

$$\frac{1}{r}\left[\frac{\partial}{\partial x}(urf) + \frac{\partial}{\partial r}(vrf) - \frac{\partial}{\partial x}\left(r\Gamma_f \frac{\partial f}{\partial x}\right) - \frac{\partial}{\partial r}\left(r\Gamma_f \frac{\partial f}{\partial r}\right)\right] = S_f. \quad (1)$$

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